

ASYMPTOTIC PLATEAU PROBLEM IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. We give a fairly complete solution to the asymptotic Plateau Problem for area minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In particular, we identify the collection of Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which bounds an area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, we study the similar problem for minimal surfaces, and show that the situation is highly different.

1. INTRODUCTION

We will call a finite collection of disjoint Jordan curves Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *fillable*, if Γ bounds a complete, embedded minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty S = \Gamma$. We will call Γ *strongly fillable* if Γ bounds a complete, embedded, area minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. In this paper, we study the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$. In particular, our aim is to classify fillable and strongly fillable infinite curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.

In the last decade, minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ have been studied extensively, and many important results have been obtained on the existence of many different types of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and their properties, e.g. [NR, CR, MMR, MoR, MRR, PR, ST1, ST2].

Recently, Kloeckner and Mazzeo [KM], and the author [Co1] independently studied the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$. While the author gave a fairly complete classification of strongly fillable finite curves in [Co1], Kloeckner and Mazzeo constructed many interesting families of finite and infinite fillable curves in [KM]. In this paper, we study the infinite curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, and complete the classification of strongly fillable curves. Our main result is as follows:

Theorem 1.1. *Let Γ be a finite collection of disjoint Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Γ is strongly fillable if and only if Γ is fat at infinity, and tall.*

Let $\Gamma^\pm = \Gamma \cap (\overline{\mathbb{H}^2} \times \{\pm\infty\})$ and $\tilde{\Gamma} = \Gamma - (\Gamma^+ \cup \Gamma^-)$. We call Γ *infinite* if either Γ^+ or Γ^- is nonempty, and *finite* otherwise. For a given Γ , being fat at infinity is completely determined by Γ^\pm , while being tall is determined solely by $\tilde{\Gamma}$. In [Co1], we introduced the notion of being tall, and showed that a finite curve Γ is strongly fillable if and only if Γ is tall. Here, we

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introduce a new notion for infinite curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, namely being *fat* or *skinny at infinity* which characterizes the behavior of Γ^+ and Γ^- .

The outline of the paper is as follows. The main difference for the solution to the asymptotic Plateau problem for \mathbb{H}^3 and for $\mathbb{H}^2 \times \mathbb{R}$ is the *escaping to infinity* problem. In \mathbb{H}^3 , a sequence of compact area minimizing surfaces $\{\Sigma_n\}$ with $\partial \Sigma_n \rightarrow \Gamma$ limits to an area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$ since convex hull of Γ is a natural barrier for the sequence $\{\Sigma_n\}$ [An]. However, in $\mathbb{H}^2 \times \mathbb{R}$, such a sequence might escape to the infinity as discussed in [Co1]. Hence, for the existence part of the main result, we first build a barrier near infinity to cover Γ^c in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ so that no piece of the sequence Σ_n escapes to infinity, and the limit area minimizing surface Σ has the desired asymptotic boundary, i.e. $\partial_\infty \Sigma = \Gamma$. To build the barrier near infinity, we use finite tall rectangles for the cylinder $S_\infty^1 \times \mathbb{R}$, and infinite rectangles and Scherk graphs (see section 2.3) for the caps $\mathbb{H}^2 \times \{\pm\infty\}$. For the nonexistence direction, we use the Scherk graphs and area comparison to get a contradiction.

The organization of the paper is as follows. In the next section, we give some definitions and related results. In Section 3, we give the complete solution to the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$, and prove Theorem 3.1. In Section 4, we discuss fillable and nonfillable infinite curves in $\mathbb{H}^2 \times \mathbb{R}$. Finally in section 5, we give some concluding remarks on further generalizations and directions.

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2. PRELIMINARIES

In this section, we will give the basic definitions, and the past results which will be used in the paper.

Throughout the paper, we will use the product compactification of $\mathbb{H}^2 \times \mathbb{R}$. In particular, $\overline{\mathbb{H}^2 \times \mathbb{R}} = \mathbb{H}^2 \times \mathbb{R} \cup \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ where $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ consists of three components, i.e. the infinite cylinder $S_\infty^1 \times \mathbb{R}$ and the caps at infinity $\overline{\mathbb{H}^2} \times \{+\infty\}$, $\overline{\mathbb{H}^2} \times \{-\infty\}$. Informally, $\overline{\mathbb{H}^2 \times \mathbb{R}}$ looks like a solid cylinder under this compactification.

Definition 2.1. A surface is *minimal* if the mean curvature H vanishes everywhere. A compact surface with boundary Σ is called *area minimizing surface* if Σ has the smallest area among the surfaces with the same boundary. A noncompact surface is called *area minimizing surface* if any compact subsurface is an area minimizing surface. Note that any area minimizing surface is minimal.

In this paper, we aim to classify the curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which bounds a complete embedded minimal (or area minimizing) surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Convention: [Curve] By *curve*, we mean a finite collection of disjoint Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ throughout the paper unless otherwise stated.

Definition 2.2 (Fillable Curves). Let Γ be a curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. We will call Γ *fillable* if Γ bounds a complete embedded minimal surface S in $\mathbb{H}^2 \times \mathbb{R}$, i.e. $\partial_\infty S = \Gamma$. We will call Γ *strongly fillable* if Γ bounds a complete embedded area minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$, i.e. $\partial_\infty \Sigma = \Gamma$. We call such S or Σ as *filling surface* for Γ .

Notice that a strongly fillable curve is fillable. Note also that fillable curves corresponds to *minimally fillable curves* in [KM].

The Asymptotic Plateau Problem for $\mathbb{H}^2 \times \mathbb{R}$ is the following classification problems:

Which Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is fillable or strongly fillable?

In order to describe the past results, we need some definitions. Throughout the paper, we will use the following notation for the curves at infinity. $\Gamma = \Gamma^+ \cup \Gamma^- \cup \tilde{\Gamma}$ where $\Gamma^\pm = \Gamma \cap (\overline{\mathbb{H}^2} \times \{\pm\infty\})$ and $\tilde{\Gamma} = \Gamma \cap (S_\infty^1 \times \mathbb{R})$. In particular, Γ^\pm is a collection of closed arcs in the caps at infinity, where $\tilde{\Gamma}$ is a collection of open arcs and closed curves in the infinite cylinder.

With this notation, we will call a curve Γ *finite* if $\Gamma^+ = \Gamma^- = \emptyset$. We will call Γ *infinite* otherwise.

2.1. Finite Curves.

When Γ is an essential Jordan curve in $S_\infty^1 \times \mathbb{R}$ which is a vertical graph over $S_\infty^1 \times \{0\}$, then there exists a vertical graph over $\mathbb{H}^2 \times \{0\}$ giving a positive answer to this existence question [NR]. However, for some null-homotopic simple closed curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, the situation can be quite different. Unlike the \mathbb{H}^3 case [An], Sa Earp and Toubiana proved that there are some nonfillable Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ [ST1].

Definition 2.3. [Thin tail] Let Γ be a simple closed curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, and let γ be an arc in Γ . Assume that there is a vertical straight line L_0 in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ such that

- $\gamma \cap L_0 \neq \emptyset$ and $\partial\gamma \cap L_0 = \emptyset$,
- γ stays in one side of L_0 ,
- $\gamma \subset \partial_\infty \mathbb{H}^2 \times (c, c + \pi)$ for some $c \in \mathbb{R}$.

Then, we call γ a *thin tail* in Γ .

Lemma 2.4. [ST1] *Let Γ be a simple closed curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. If Γ contains a thin tail, then there is no properly immersed minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma \supset \Gamma$.*

The above result shows that the curves with thin tail cannot be fillable. Hence, to bypass this obstruction, we introduced the following notion.

Definition 2.5. [Tall Curves] [Co1] Consider $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with the coordinates (θ, t) where $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$. We will call the rectangle $R = [\theta_1, \theta_2] \times [t_1, t_2] \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ as *tall rectangle* if $t_2 - t_1 > \pi$.

We call a curve Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *tall curve* if the open region $\Gamma^c \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ can be written as a union of tall rectangles $\text{int}(R_i)$, i.e. $\Gamma^c = \bigcup_i \text{int}(R_i)$.

Notice that tall curves do not have thin tails. Furthermore, we define the *height of a curve*, $h(\Gamma)$, as the length of the smallest vertical line segment in Γ^c . Hence, Γ is tall if and only if $h(\Gamma) > \pi$.

In [Co1], we gave a fairly complete classification of strongly fillable finite curves as follows.

Lemma 2.6. [Co1] *Let Γ be a finite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with $h(\Gamma) \neq \pi$. Then, Γ is strongly fillable if and only if Γ is a tall curve.*

After this result for finite curves, we aim to give a characterization for *infinite* strongly fillable curves to complete the classification.

2.2. Infinite Curves.

The key property of infinite fillable curves is given in [KM, Proposition 4.3].

Lemma 2.7. [KM] *If Γ is infinite fillable curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, then Γ^\pm must be a collection of geodesics in $\mathbb{H}^2 \times \{\pm\infty\}$.*

Now, we give a generalization of tall rectangles to infinite curves.

Definition 2.8. [ST1] [Infinite Rectangles] Let γ be a complete geodesic in \mathbb{H}^2 with $\partial_\infty \gamma = \{p, q\}$. Let α be one of the two arcs in $S_\infty^1(\mathbb{H}^2)$ with endpoints p and q . Fix $t_0 \in \mathbb{R}$. Let $l_p^+ = \{p\} \times [t_0, \infty]$ and $l_p^- = \{p\} \times [-\infty, t_0]$ be the vertical line segments in $S_\infty^1 \times \mathbb{R}$. Let $\gamma^\pm = \gamma \times \{\pm\infty\}$ be the geodesic in $\mathbb{H}^2 \times \{\pm\infty\}$. Let $\alpha_0 = \alpha \times \{t_0\}$. Then define $\mathcal{R}^+ = \gamma^+ \cup l_p^+ \cup l_q^+ \cup \alpha_0$ is an infinite rectangle. Similarly, $\mathcal{R}^- = \gamma^- \cup l_p^- \cup l_q^- \cup \alpha_0$ is also an infinite rectangle. See Figure 6-Left, where each component is an infinite rectangle.

Lemma 2.9. *Any infinite rectangle in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is strongly fillable, and it bounds a unique area minimizing surface in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.*

Proof: Let $\mathcal{R} = \gamma^+ \cup l_p^+ \cup l_q^+ \cup \alpha_0$ be an infinite rectangle with the notation above. By [ST1], \mathcal{R} bounds a minimal surface \mathcal{T} in $\mathbb{H}^2 \times \mathbb{R}$, i.e. $\partial_\infty \mathcal{T} = \mathcal{R}$. Furthermore, \mathcal{T} is a graph over the region Δ in $\overline{\mathbb{H}^2}$ separated by $\gamma \cup \alpha$ [ST1].

Consider the family of minimal surfaces $\{\mathcal{T}_s \mid s \in \mathbb{R}\}$ where \mathcal{T}_s is s vertical translation of \mathcal{T} . By construction, $\{\mathcal{T}_s\}$ foliates the convex region $\Delta \times \mathbb{R}$. This shows that \mathcal{T} is area minimizing, and the unique minimal surface \mathcal{R} bounds in $\mathbb{H}^2 \times \mathbb{R}$. Hence, \mathcal{R} is strongly fillable. \square

2.3. Scherk Graphs.

Now, we recall the results on Scherk graphs in $\mathbb{H}^2 \times \mathbb{R}$ by [CR]. These are minimal graphs over ideal $2n$ -gons in \mathbb{H}^2 where the graph takes values $+\infty$ and $-\infty$ on alternating sides. In particular, let Δ be an ideal $2n$ -gon in \mathbb{H}^2 . Let $\mathcal{V} = \{p_1, p_2, \dots, p_{2n}\} \in S_\infty^1(\mathbb{H}^2)$ be the set of ideal vertices of Δ which are circularly ordered. Let α_i be the geodesic with $\partial_\infty \alpha_i = \{p_{2i-1}, p_{2i}\}$ and β_i be the geodesic with $\partial_\infty \beta_i = \{p_{2i}, p_{2i+1}\}$. Then, $\partial\Delta = \alpha_1 \cup \beta_1 \cup \dots \cup \alpha_n \cup \beta_n$. For each ideal vertex p_i , define a sufficiently small horocycle C_i such that $C_i \cap C_j = \emptyset$ for any $1 \leq i < j \leq 2n$. Let B_i be the horodisk which C_i bounds in \mathbb{H}^2 . Let $\Omega = \mathbb{H}^2 - \bigcup_i B_i$. Let $\hat{\alpha}_i = \alpha_i \cap \Omega$ and $\hat{\beta}_i = \beta_i \cap \Omega$. Let $a(\Delta) = \sum |\hat{\alpha}_i|$ and $b(\Delta) = \sum |\hat{\beta}_i|$. See Figure 1-left.

We say an ideal polygon \mathcal{D} is *inscribed in Δ* if $\partial_\infty \mathcal{D} \subset \mathcal{V}$. Clearly, $\partial\mathcal{D}$ consists of some geodesics in $\partial\Delta$, and some other geodesics $\{\gamma_j\}$ in the interior of Δ . Let $a(\mathcal{D})$ is the sum of $|\hat{\alpha}_i|$ where $\alpha_i \subset \partial\mathcal{D}$, and similarly define $b(\mathcal{D})$. Let $c(\mathcal{D}) = \sum |\hat{\gamma}_j|$ where $\hat{\gamma}_j = \gamma_j \cap \Omega$. Then, let $|\mathcal{D}|$ be the sum of the "truncated lengths" of the geodesics in $\partial\mathcal{D}$, i.e. $|\mathcal{D}| = a(\mathcal{D}) + b(\mathcal{D}) + c(\mathcal{D})$.

Definition 2.10 (Exact Polygons). Let Δ be an ideal $2n$ -gon in \mathbb{H}^2 . For any inscribed polygon \mathcal{D} in Δ , let $2a(\mathcal{D}) < |\mathcal{D}|$ and $2b(\mathcal{D}) < |\mathcal{D}|$. Then, we call Δ a *regular ideal polygon*.

Let Δ be a regular ideal polygon with $a(\Delta) = b(\Delta)$. Then, we call Δ an *exact ideal polygon*.

Recently, Collin and Rosenberg showed the existence of solutions to the Dirichlet problem with $\pm\infty$ boundary values for exact ideal polygons [CR, Theorem 1].

Lemma 2.11. [CR] [Scherk Graphs] *Let Δ be an exact ideal $2n$ -gon in \mathbb{H}^2 . Then, there exists a solution $u : \Delta \rightarrow \mathbb{R}$ to the minimal surface equation on Δ which takes values $+\infty$ on α_i and $-\infty$ on β_i for $1 \leq i \leq n$. Furthermore, the solution is unique up to an additive constant.*

Remark 2.12. [Area Minimizing] The Scherk graph $\Sigma = \text{graph}(u)$ is a minimal surface in \mathbb{H}^2 with

$$\xi = \partial_\infty \Sigma = \bigcup_{i=1}^n (\alpha_i \times \{+\infty\}) \cup (\beta_i \times \{-\infty\}) \bigcup_{j=1}^{2n} l_{p_j}$$

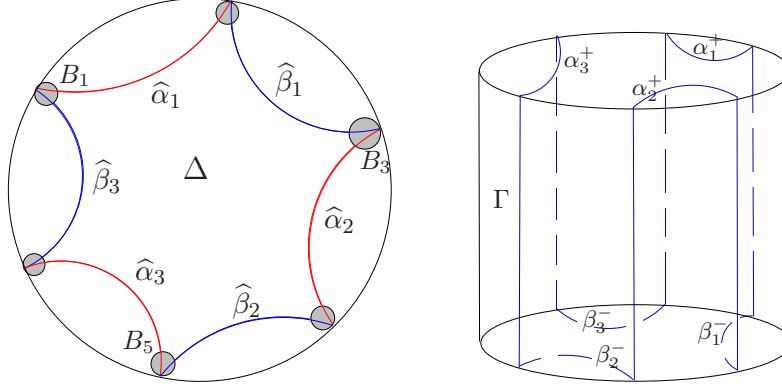


FIGURE 1. In the figure left, Δ is an ideal hexagon where B_i represents the small horoballs at $p_i \in S_\infty^1$. In the right, the asymptotic boundary of a Scherk graph is given.

where l_p is the vertical line $\{p\} \times \mathbb{R}$ in $S_\infty^1 \times \mathbb{R}$. We will call the asymptotic boundary ξ of a Scherk graph a *Scherk curve* in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. See Figure 1-right.

Notice that Σ is also an area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ because the family of surfaces $\{\Sigma_t \mid t \in \mathbb{R}\}$ foliates the convex region $\Delta \times \mathbb{R}$ where Σ_t is the t vertical translation of Σ .

2.4. Fat / Skinny at Infinity.

Now, we introduce a new notion to study the fillability of the infinite curves by using Scherk graphs. Let $\Gamma = \Gamma^+ \cup \Gamma^- \cup \tilde{\Gamma}$ as before. Let $\Gamma^+ = \gamma_1^+ \cup \dots \gamma_n^+$ and $\Gamma^- = \gamma_1^- \cup \dots \gamma_m^-$ for $n, m > 1$. $n = 1$ or $m = 1$ cases are trivial, and they will be discussed later. As Γ is a finite collection of disjoint Jordan curves, no endpoints of γ_i^+ and γ_j^+ are same for $i \neq j$. Let $\mathcal{V}^+ = \bigcup \partial_\infty \gamma_i^+$ be the set of $2n$ points in $S_\infty^1(\mathbb{H}^2) \times \{+\infty\}$. Let $\mathcal{V}^+ = \{p_1^+, p_2^+, \dots, p_{2n}^+\}$ is indexed so that the points are circularly ordered. For $1 \leq i \leq 2n$, let τ_i^+ be the geodesic in $\mathbb{H}^2 \times \{+\infty\}$ with $\partial_\infty \tau_i^+ = \{p_i^+, p_{i+1}^+\}$. Of course, for some i , $\tau_i^+ \subset \Gamma^+$. Similarly, define \mathcal{V}^- be the set of endpoints of γ_j^- , and τ_j^- be the geodesics between them for $1 \leq j \leq 2m$.

Let Δ^+ be the convex hull of \mathcal{V}^+ in $\mathbb{H}^2 \times \{+\infty\}$. In particular, Δ^+ is the ideal $2n$ -gon in $\mathbb{H}^2 \times \{+\infty\}$ with $\partial \Delta^+ = \bigcup_{i=1}^{2n} \tau_i^+$. Similarly define Δ^- which is ideal $2m$ -gon with $\partial \Delta^- = \bigcup_{j=1}^{2m} \tau_j^-$. Notice that some of the geodesics γ_i^+ might be in the interior of Δ^+ . Hence, $\Delta^+ - \Gamma^+$ is a union of ideal polygons in $\mathbb{H}^2 \times \{+\infty\}$, i.e. $\Delta^+ - \Gamma^+ = \Delta_1^+ \cup \dots \cup \Delta_{n_1}^+$ for some $n_1 \geq 1$, where the vertices of Δ_i^+ is in \mathcal{V} (See Figure 2-right). Similarly, let $\Delta^- - \Gamma^- = \Delta_1^- \cup \dots \Delta_{n_2}^-$. By abuse of notation, we will take Δ_i^\pm as

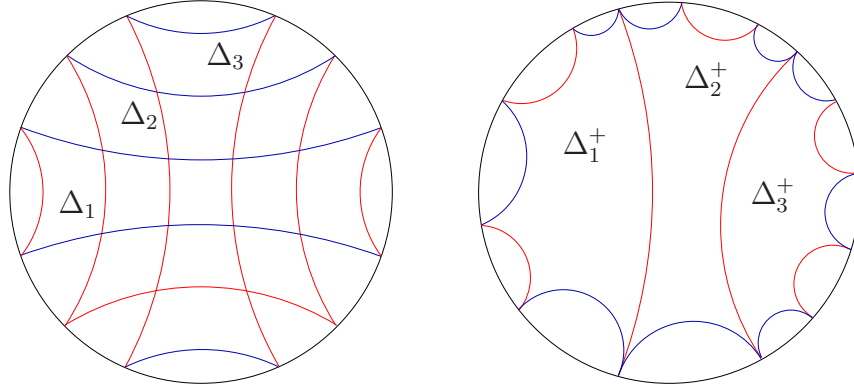


FIGURE 2. In the figure left, Δ_1, Δ_2 and Δ_3 represent a fat, exact, and skinny polygon respectively. Red curves represent α curves, and blue curves represent β curves in $\partial\Delta_i$. In the figure right, 7 red geodesics represent Γ^+ , and Δ^+ decomposed into 3 inscribed polygons $\Delta^+ = \Delta_1^+ \cup \Delta_2^+ \cup \Delta_3^+$.

the closed polygons containing its sides, i.e. $\Delta^+ = \Delta_1^+ \cup \dots \cup \Delta_{n_1}^+$ and $\Delta^- = \Delta_1^- \cup \dots \cup \Delta_{n_2}^-$. Notice that each Δ_i^\pm is inscribed in Δ^\pm .

In particular, Δ^\pm naturally decomposes as a union of *inscribed* ideal polygons Δ_i^\pm . Notice that each Δ_i^\pm contains k_i^\pm geodesics from Γ^\pm by construction, and hence, Δ_i^\pm is $2k_i^\pm$ -gon for some $k_i^\pm > 1$. Furthermore, the geodesics in Γ^+ alternates in $\partial\Delta_i^+$, and similarly Γ^- in Δ_i^- . Note also that if Δ^\pm is regular, then Δ_i^\pm is regular as Δ_i^\pm is inscribed in Δ^\pm .

Definition 2.13 (Fat / Skinny at Infinity). For $n > 1$, let Ω be a regular ideal $2n$ -gon in \mathbb{H}^2 with $\partial\Omega = \alpha_1 \cup \beta_1 \cup \dots \cup \alpha_n \cup \beta_n$ where the geodesics α_i and β_i are alternating. Let $a(\cdot)$ and $b(\cdot)$ be as defined in Scherk graphs section. We will call Ω *fat* if $a(\Omega) < b(\Omega)$. We will call Ω *skinny* if $a(\Omega) > b(\Omega)$. See Figure 2-left.

Let Γ be an infinite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Define Δ^\pm as above. Let $\Delta^+ = \Delta_1^+ \cup \dots \cup \Delta_{n_1}^+$ and $\Delta^- = \Delta_1^- \cup \dots \cup \Delta_{n_2}^-$ be induced decompositions of Δ^\pm as defined before. Let the geodesics in $\Gamma^+ \cap \partial\Delta_i^+$ be α -curves of Δ_i^+ , and let the geodesics in $\Gamma^- \cap \partial\Delta_j^-$ be again α -curves of Δ_j^- . Then, we will call Γ *fat at infinity* if all ideal polygons Δ_i^+ and Δ_j^- are fat. Furthermore, we will call Γ *skinny at infinity* if at least one polygon Δ_i^\pm is skinny.

Note that if $n = 1$ or $m = 1$ (number of components in Γ^\pm), we still call Γ is fat at infinity, and the proofs in Section 3 applies to this case trivially.

The following lemma implies that any fat polygon can be covered by a finite union of exact polygons. See Figure 3. In particular, let Ω be a fat $2n$ -gon with the vertices $\mathcal{V} = \{p_1, \dots, p_{2n}\}$ which is circularly ordered. Hence,

Ω is the convex hull of \mathcal{V} in \mathbb{H}^2 . Define $p_{2n+1} = p_1$. Let $\alpha_i = \overrightarrow{p_{2i-1}p_{2i}}$ and $\beta_i = \overrightarrow{p_{2i}p_{2i+1}}$ where \overrightarrow{pq} represents the geodesic between p and q . Hence, $\partial\Omega = \alpha_1 \cup \beta_1 \cup \dots \cup \alpha_n \cup \beta_n$ where the geodesics α_i and β_i are alternating.

Lemma 2.14. *[Fats covered by Exacts] Let Ω be a fat $2n$ -gon as above. Then, Ω can be covered by exact $2n$ -gons \mathcal{D}_i where $\mathcal{D}_i \cap \alpha_j = \emptyset$ for any i, j , i.e. $\Omega \subset \bigcup_i \mathcal{D}_i$.*

Proof: For $p, q \in S_\infty^1(\mathbb{H}^2)$, let $[p, q]$ represents the interval from p to q in the counterclockwise direction. We represent circularly order of \mathcal{V} with $p_1 \prec p_2 \prec \dots \prec p_{2n} \prec p_1$. Let \mathcal{D} be an ideal $2n$ -gon with vertices $\mathcal{W} = \{q_1, q_2, \dots, q_{2n}\}$ circularly ordered. In addition, we assume that $[q_{2i}, q_{2i+1}] \subset [p_{2i}, p_{2i+1}]$. In other words, $q_1 \prec p_1 \prec p_2 \prec q_2 \prec q_3 \prec p_3 \prec \dots \prec p_{2n} \prec q_{2n} \prec q_1$. Hence $\mathcal{D} \cap \alpha_i = \emptyset$ for any $1 \leq i \leq n$. Furthermore, this implies $a(\Omega) < a(\mathcal{D})$ and $b(\Omega) > b(\mathcal{D})$ by construction.

Let $\mathcal{W}_i = \{p_1, p_2, \dots, p_{i-1}, p_i^*, p_{i+1}, \dots, p_{2n}\}$ be the set vertices of \mathcal{D}_i such that $\mathcal{V} \triangle \mathcal{W}_i = \{p_i, p_i^*\}$ and \mathcal{D}_i is exact. Here, if i is odd, we choose p_i^* in (p_{i-1}, p_i) and if i is even, we choose p_i^* in (p_i, p_{i+1}) . We claim we can always find a unique p_i^* in the given interval so that \mathcal{D}_i is exact.

In particular, for $i = 2k$, \mathcal{D}_i has the same geodesics with Ω except α_k and β_k . Furthermore, for $i = 2k$, if $p_i^* \rightarrow p_{i+1}$, then β_k escapes to infinity, and the quantity $b(\mathcal{D}_i) - a(\mathcal{D}_i) \searrow -\infty$ monotonically as $p_i^* \rightarrow p_{i+1}$. Also, if p_i^* moves to other direction, $p_i^* \rightarrow p_i$, then $\mathcal{D}_i \rightarrow \Omega$ and $b(\mathcal{D}_i) - a(\mathcal{D}_i) \nearrow (b(\Omega) - a(\Omega)) > 0$ monotonically. This proves that there exists a unique point $p_i^* \in (p_i, p_{i+1})$ with $a(\mathcal{D}_i) = b(\mathcal{D}_i)$.

For $i = 2k + 1$, \mathcal{D}_i has the same geodesics with Ω except α_{k+1} and β_k . Similar to the even case, if $i = 2k + 1$, when $p_i^* \rightarrow p_{i-1}$, β_k escapes to

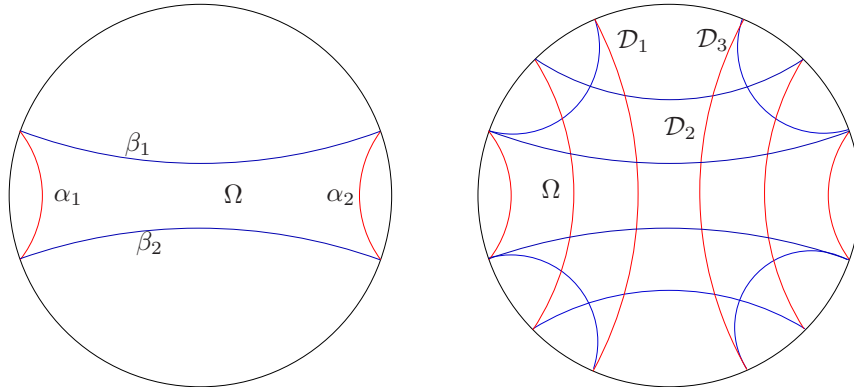


FIGURE 3. [Exact Covering] In the figure left, Ω represents a fat 4-gon. In the figure right, Ω is covered by 3 exact 4-gons, $\mathcal{D}_1, \mathcal{D}_2$, and \mathcal{D}_3 , i.e. $\Omega \subset \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$.

infinity, and the quantity $b(\mathcal{D}_i) - a(\mathcal{D}_i) \searrow -\infty$ monotonically. In the other direction, if $p_i^* \rightarrow p_i$, then $\mathcal{D}_i \rightarrow \Omega$ and $b(\mathcal{D}_i) - a(\mathcal{D}_i) \nearrow (b(\Omega) - a(\Omega)) > 0$ monotonically again. Hence, there exists a unique point $p_i^* \in (p_{i-1}, p_i)$ with $a(\mathcal{D}_i) = b(\mathcal{D}_i)$ in this case, too.

For $1 \leq i \leq 2n$, define \mathcal{D}_i as described above. Then \mathcal{D}_i is an exact $2n$ -gon which has the same sides with Ω except β_k and α_k for $i = 2k$ (α_{k+1} for $i = 2k + 1$). Then, it is not hard to show that $\Omega \subset \bigcup_{i=1}^{2n} \mathcal{D}_i$. Let $\mathcal{V}_i = \mathcal{V} - \{p_i\}$ for $1 \leq i \leq 2n$. $\tilde{\mathcal{D}}_i$ be the ideal $(2n - 1)$ -gon with vertices \mathcal{V}_i . As $\mathcal{V}_i \subset \mathcal{W}_i$, then $\tilde{\mathcal{D}}_i \subset \mathcal{D}_i$. In particular, $\tilde{\mathcal{D}}_i$ is obtained by removing the ideal triangle Δ_i with vertices $\{p_{i-1}, p_i, p_{i+1}\}$ from Ω , i.e. $\tilde{\mathcal{D}}_i = \Omega - \Delta_i$. Since $\tilde{\mathcal{D}}_i \subset \mathcal{D}_i$ for any $1 \leq i \leq 2n$, it is clear that $\Omega \subset \bigcup_{i=1}^{2n} \mathcal{D}_i$, and the proof follows. \square

We will call such coverings of fat $2n$ -gon Ω by exact $2n$ -gons $\{\mathcal{D}_i\}$ as in the lemma, an *exact covering of Ω* . See Figure 3.

Remark 2.15 (Exceptional Curves). Throughout the paper, we will not consider the infinite curves which are neither fat nor skinny. These are the curves which contains no skinny polygon at infinity, and an ideal polygon Δ_i^\pm with $a(\Omega) = b(\Omega)$. After ignoring these exceptional curves, if an infinite curve is not fat, it must be skinny.

3. CLASSIFICATION OF STRONGLY FILLABLE CURVES

Let Γ be an infinite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Let $\Gamma^\pm = \Gamma \cap (\overline{\mathbb{H}^2} \times \{\pm\infty\})$ and $\tilde{\Gamma} = \Gamma - (\Gamma^+ \cup \Gamma^-)$ as before. Recall that Γ^\pm is a collection of disjoint geodesics in $\mathbb{H}^2 \times \{\pm\infty\}$ by Lemma 2.7. In this section, we will study the role of Γ^\pm on Γ for strong fillability, and show that it completely determines strong fillability for a given infinite tall curve Γ .

On the other hand, we will study the fillability question in the next section, and see that the fillability question and the strong fillability question are quite different for infinite curves. While Γ^\pm completely determines strong fillability for a given infinite tall curve Γ , it is not very useful to detect fillability (See Section 4.1).

Now, we will prove the main theorem. Recall that by curve, we mean a finite collection of disjoint Jordan curves. Note also our convention for exceptional curves (Remark 2.15).

Theorem 3.1. *Let Γ be a tall infinite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Then, Γ is strongly fillable if and only if Γ is fat at infinity.*

Proof: We will divide the proof into two parts: Existence for fat curves, and nonexistence for skinny curves.

Step 1: If Γ is fat at infinity then Γ is strongly fillable.

Proof of Step 1: We will show that there exists an area minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. First, we start by constructing a sequence of compact area minimizing surfaces Σ_n with $\partial \Sigma_n \rightarrow \Gamma$. Our aim is to take the limit of Σ_n , and to show that the limit area minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ has the asymptotic boundary Γ . However, as indicated in [Co1], the sequence might escape to infinity. Then, we might end up with an empty limit, or a nonempty limit Σ with $\partial_\infty \Sigma \subset \Gamma$ but $\partial_\infty \Sigma \neq \Gamma$. Hence, to overcome this problem, we need to construct a barrier \mathcal{N} in $\mathbb{H}^2 \times \mathbb{R}$ (See also [KM, Proposition 4.1]).

Step 1a: The construction of the barrier near infinity.

The barrier \mathcal{N} can be considered as a neighborhood of $\Gamma^c = (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) - \Gamma$ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$ such that \mathcal{N}^c is mean convex, $\partial_\infty \mathcal{N}^c = \Gamma$ and $\Sigma_n \subset \mathcal{N}^c$. In particular, we want to keep the sequence $\{\Sigma_n\}$ away from $\partial_\infty \mathbb{H}^2 \times \mathbb{R} - \Gamma$ in order to prevent Σ_n escape to infinity, i.e. $\Sigma_n \subset \mathcal{N}^c$ and $\partial_\infty \mathcal{N}^c = \Gamma$. Hence, the condition $\mathcal{N} \cap \Sigma_n = \emptyset$ makes sure this, and \mathcal{N} would act as a barrier between Σ_n and $\partial_\infty \mathbb{H}^2 \times \mathbb{R} - \Gamma$. In [Co1], we constructed such a barrier at infinity for finite curves. Now, we construct such a barrier for infinite curves.

We will use the notation above. To construct the barrier near the cylinder $S_\infty^1 \times \mathbb{R}$, we can use the tall rectangles again as in [Co1]. Hence, the main problem is the to construct a barrier near the caps at infinity $\mathbb{H}^2 \times \{\pm\infty\}$ to prevent the sequence escape to infinity.

Γ is a finite collection of disjoint Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with $\Gamma = \Gamma^+ \cup \Gamma^- \cup \tilde{\Gamma}$. Since Γ is infinite curve, either Γ^+ or Γ^- is nonempty. Without loss of generality assume $\Gamma^+ = \gamma_1^+ \cup \dots \cup \gamma_n^+$, and $\Gamma^- = \gamma_1^- \cup \dots \cup \gamma_m^-$. Define Δ^\pm , τ_i^+ and τ_j^- as in Section 2.4 where $1 \leq i \leq 2n$ and $1 \leq j \leq 2m$. We will assume $m, n > 1$. $n = 1$ or $m = 1$ cases are trivial, and will be explained later.

The barrier \mathcal{N} consists of three major blocks: Infinite side barriers, Scherk barriers at infinity, and the tall rectangles. The tall rectangles are constructed in [Co1], and it covers $\tilde{\Gamma}^c$ in the cylinder $S_\infty^1 \times \mathbb{R}$. The Scherk barriers covers the inside of Δ^\pm in $\mathbb{H}^2 \times \{\pm\infty\}$, while infinite side barriers covers the outside of Δ^\pm in $\mathbb{H}^2 \times \{\pm\infty\}$.

Infinite side barriers: For each τ_i^+ , we define an infinite side barrier $\hat{\mathcal{T}}_i^+$ as follows. Recall that $\partial_\infty \tau_i^+ = \{p_i^+, p_{i+1}^+\}$. Let $p_i^+ = (\theta_i, +\infty)$ for $1 \leq i \leq 2n$ where $\theta_i \in [0, 2\pi)$. If there exists an infinite rectangle \mathcal{R}_i^+ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ (Definition 2.8) such that $\mathcal{R}_i^+ \cap \Gamma = \tau_i^+$, then let \mathcal{T}_i^+ be the unique area minimizing surface \mathcal{R}_i^+ bounds in $\mathbb{H}^2 \times \mathbb{R}$ by Lemma 2.9. Let

$\widehat{\mathcal{T}}_i^+$ be the domain separated by \mathcal{T}_i^+ from $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \widehat{\mathcal{T}}_i^+$ contains the arc $(p_i, p_{i+1}) \times \{+\infty\}$ in the upper corner of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.

If there is no such \mathcal{R}_i^+ , it means that near $\partial_\infty \tau_i^+$, $\tilde{\Gamma}$ is a graph over the arcs $\{p_i\} \times (N_i, +\infty)$ for sufficiently large N_i . Let $\eta_i^+ \cup \eta_{i+1}^+ = \Gamma \cap (\theta_i, \theta_{i+1}) \times [N_i, \infty)$. Let $u_i^+ : (N_i, \infty) \rightarrow [\theta_i, \theta_{i+1})$ and $u_{i+1}^+ : (N_i, \infty) \rightarrow (\theta_i, \theta_{i+1})$ be maps such that $\eta_i^+ = \text{graph}(u_i^+)$ and $\eta_{i+1}^+ = \text{graph}(u_{i+1}^+)$. Choose N_i sufficiently large so that u_i^+ is monotone decreasing, and u_{i+1}^+ is monotone increasing map. Note that $u_i^+(t) \rightarrow \theta_i$ and $u_{i+1}^+(t) \rightarrow \theta_{i+1}$ as $t \nearrow \infty$. For this $N_i \gg 0$, if η_i^+ or η_{i+1}^+ is empty, define the corresponding $u_i^+(t) = \theta_i$ or $u_{i+1}^+(t) = \theta_{i+1}$ constant maps. Then, $\eta_i^+ = \{(u_i^+(t), t) \in S_\infty^1 \times \mathbb{R} \mid t \in (N_i, \infty)\}$. See Figure 4-left.

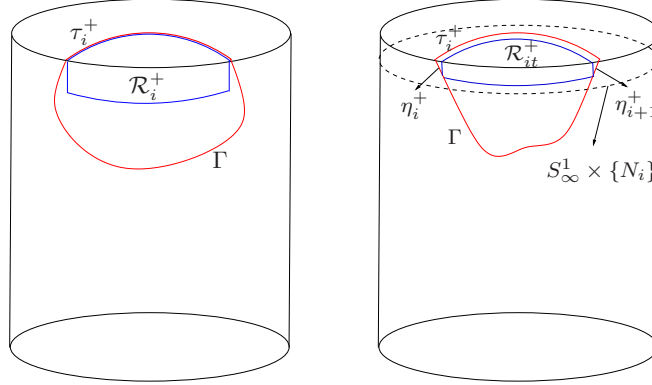


FIGURE 4. In the left, we have the trivial case, where we can cover outside of τ_i^+ by one infinite rectangle \mathcal{R}_i^+ as $\tilde{\Gamma}$ curves outside. In the right, we need to use a family of infinite rectangles $\{\mathcal{R}_{it}^+\}$ to cover as $\tilde{\Gamma}$ curves inside.

For $t > N_i$, define \mathcal{R}_{it}^+ be the infinite rectangle containing $\partial([u_i^+(t), u_{i+1}^+(t)] \times [t, +\infty))$ and the geodesic τ_{it}^+ in $\mathbb{H}^2 \times \{+\infty\}$ with $\partial_\infty \tau_{it}^+ = \{(u_i^+(t), \infty), (u_{i+1}^+(t), \infty)\}$. See Figure 4-right. Let \mathcal{T}_{it}^+ be the unique area minimizing surface bounding \mathcal{R}_{it}^+ . Let $\widehat{\mathcal{T}}_{it}^+$ be the domain separated by \mathcal{T}_{it}^+ from $\mathbb{H}^2 \times \mathbb{R}$ where $\partial_\infty \widehat{\mathcal{T}}_{it}^+$ contains the arc $(u_i^+(t), u_{i+1}^+(t)) \times \{+\infty\}$ in the upper corner of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Then define $\widehat{\mathcal{T}}_i^+ = \bigcup_{t > N_i} \widehat{\mathcal{T}}_{it}^+$. Notice that $\partial_\infty \widehat{\mathcal{T}}_i^+ \cap \mathbb{H}^2 \times \{+\infty\}$ is the region bounded by τ_i^+ . Furthermore, $\partial(\partial_\infty \widehat{\mathcal{T}}_i^+) \cap \tilde{\Gamma} = \eta_i^+ \cup \eta_{i+1}^+$ by construction. Similarly define $\widehat{\mathcal{T}}_j^-$ for $1 \leq j \leq 2m$. We will call $\widehat{\mathcal{T}}_i^\pm$ as infinite side barrier. Notice that for any τ_i^\pm , we have an infinite side barrier $\widehat{\mathcal{T}}_i^\pm$ such that $\Gamma \cap \text{int}(\partial_\infty \widehat{\mathcal{T}}_i^\pm) = \emptyset$. Hence, infinite side barriers cover outside of Δ^\pm in $\mathbb{H}^2 \times \{\pm\infty\}$.

Scherk barriers at infinity: After covering outside by infinite side barriers, we want to cover the inside of Δ^\pm to construct \mathcal{N} so that $\partial_\infty \mathcal{N}^c = \Gamma$. By Section 2.4, Δ^\pm decomposes into inscribed polygons by Γ^\pm , i.e. $\Delta^+ = \Delta_1^+ \cup \dots \Delta_{n_1}^+$ and $\Delta^- = \Delta_1^- \cup \dots \Delta_{m_1}^-$ where $n_1, m_1 \geq 1$. Since Γ is fat at infinity, by Lemma 2.14, Δ_i^\pm is a fat polygon, and it has an exact covering. In particular, for any $1 \leq i \leq n_1$, $\Delta_i^+ = \bigcup_k \mathcal{D}_{ik}^+$ where \mathcal{D}_{ik}^+ are exact ideal polygons in $\mathbb{H}^2 \times \{+\infty\}$ with $\mathcal{D}_{ik}^+ \cap \Gamma^+ = \emptyset$. See Figure 3.

Fix $\Delta_{i_o}^+$. Notice that $\Delta_{i_o}^+$ is an ideal $2n_o$ -gon for some $n_o \leq n$, and $\Delta_{i_o}^+ = \bigcup_k \mathcal{D}_{i_o k}^+$ where $\mathcal{D}_{i_o k}^+$ is an exact ideal $2n_o$ -gon. By construction, $\partial \Delta_{i_o}^+ = \gamma_{j_1} \cup \tau_{j_1} \cup \dots \cup \gamma_{j_{n_o}} \cup \tau_{j_{n_o}}$. α curves of $\Delta_{i_o}^+$ are the n_o geodesics $\gamma_{j_1}, \dots, \gamma_{j_{n_o}}$ in $\Gamma^+ \cap \partial \Delta_{i_o}^+$, while β curves of $\Delta_{i_o}^+$ are the remaining n_o geodesics $\tau_{j_1}, \dots, \tau_{j_{n_o}}$. Similarly, define α curves of $\mathcal{D}_{i_o k}^+$ as the geodesics in $\partial \mathcal{D}_{i_o k}^+$ intersecting $\Delta_{i_o}^+$, and define β curves of $\mathcal{D}_{i_o k}^+$ as the geodesics in $\partial \mathcal{D}_{i_o k}^+$ disjoint from $\Delta_{i_o}^+$.

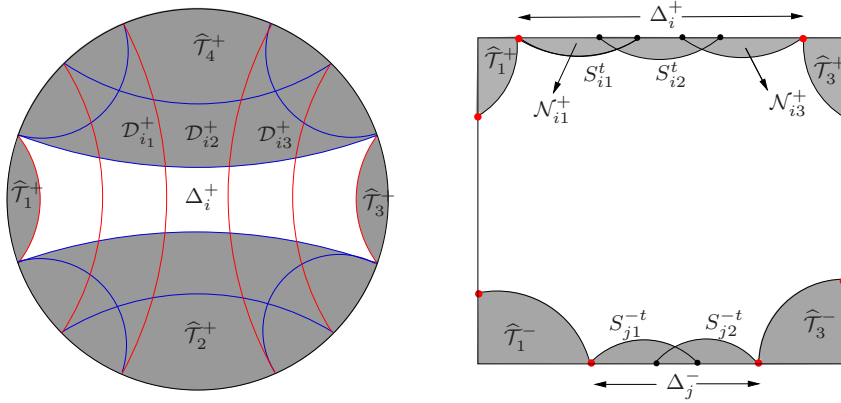


FIGURE 5. In the left, gray regions represents infinite side barriers $\hat{\tau}_i^+$, which covers outside of Δ^+ . Δ_i^+ is covered by 3 exact polygons as in Figure 3. In the right, red points represent Γ , and structure of \mathcal{N}^+ and \mathcal{N}^- are pictured from the side.

Consider the Scherk graph $S_{i_o k}$ (Lemma 2.11) over $\mathcal{D}_{i_o k}^+$ where it takes $+\infty$ value on the α -curves of $\mathcal{D}_{i_o k}^+$, and $-\infty$ value on the β -curves. Consider the infinite side barriers $\hat{\tau}_{j_i}^+$ near the β -curves of $\Delta_{i_o}^+$, i.e. $\tau_{j_1}, \dots, \tau_{j_{n_o}}$, i.e. $\partial_\infty \hat{\tau}_{j_i}^+ \supset \tau_{j_i}$.

For sufficiently large t , the Scherk graph $S_{i_o k}^t = S_{i_o k} + t$ (t vertical translation of $S_{i_o k}$) intersects $\partial \hat{\tau}_{j_i}^+$ in an arc $\sigma_{i_o k}^{j_i}$ with $\partial_\infty \sigma_{i_o k}^{j_i} \subset \tau_{j_i}$ for any j_i with $\mathcal{D}_{i_o k}^+ \cap \tau_{j_i}^+ \neq \emptyset$. Let $\hat{S}_{i_o k}^t = S_{i_o k}^t - \text{int}(\bigcup \hat{\tau}_{j_i}^+)$. Hence, $\partial \hat{S}_{i_o k}^t \supset \sigma_{i_o k}^{j_i}$. Let $\mathcal{N}_{i_o k}$ be the connected region over $\hat{S}_{i_o k}^t$ where $\partial_\infty \mathcal{N}_{i_o k} \subset \mathbb{H}^2 \times \{+\infty\}$ and $\partial \mathcal{N}_{i_o k} \subset \hat{S}_{i_o k}^t \cup \partial \hat{\tau}_{j_i}^+$. Hence, we define a barrier $\mathcal{N}_{i_o k}^+$ near the upper

cap for each exact polygon $\mathcal{D}_{i_0 k}^+$. By the construction and the shape, we call $\mathcal{N}_{i_0 k}^+$ Scherk barrier at infinity.

Recall that Δ^+ decomposes into inscribed polygons $\Delta^+ = \Delta_1^+ \cup \dots \Delta_{n_1}^+$, and each inscribed polygon covered by fat polygons, i.e. $\Delta_{i_0}^+ = \bigcup_{k=1}^{n_o} \mathcal{D}_{i_0 k}^+$. Let $\mathcal{N}_{i_0}^+ = \bigcup_{k=1}^{n_o} \mathcal{N}_{i_0 k}^+$. Notice that $\partial_\infty \mathcal{N}_{i_0}^+ = \Delta_{i_0}^+$.

Now, let $\mathcal{N}^+ = \bigcup_{i=1}^{n_1} \mathcal{N}_i^+ \bigcup_{j=1}^{2n} \widehat{\mathcal{T}}_j^+$. Then, \mathcal{N}^+ would be a neighborhood of $(\mathbb{H}^2 \times \{+\infty\}) - \Gamma^+$ in $\mathbb{H}^2 \times \mathbb{R}$. In particular, $\partial_\infty(\mathcal{N}^+)^c \cap (\mathbb{H}^2 \times \{+\infty\}) = \Gamma^+$. This means \mathcal{N}^+ can be considered as a barrier for $\mathbb{H}^2 \times \{+\infty\} - \Gamma^+$ as well as its neighborhood in $\mathbb{H}^2 \times \mathbb{R}$. Define \mathcal{N}^- similarly.

Note that if $\Gamma^+ = \emptyset$, then let $t_0^+ = \sup\{t \in \mathbb{R} \mid (\theta, t) \in \Gamma\} < \infty$ be the highest height of Γ . Then define $\mathcal{N}^+ = \mathbb{H}^2 \times (t_0, +\infty)$. Similarly, if $\Gamma^- = \emptyset$, define $\mathcal{N}^- = \mathbb{H}^2 \times (t_0^-, -\infty)$ where t_0^- is the lowest height of Γ .

Finally, if either $m = 1$ or $n = 1$, we can define the barrier \mathcal{N}^\pm without Scherk barriers. In particular, if $\Gamma^+ = \gamma^+$ is just one geodesic, then we can cover both sides of γ^+ in $\mathbb{H}^2 \times \{+\infty\}$ with infinite side barriers $\widehat{\mathcal{T}}_1^+$ and $\widehat{\mathcal{T}}_2^+$. Hence in this case, $\mathcal{N}^+ = \widehat{\mathcal{T}}_1^+ \cup \widehat{\mathcal{T}}_2^+$. If $m = 1$, \mathcal{N}^- can be defined similarly.

Step 1b: The sequence of compact area minimizing surfaces $\{\Sigma_n\}$.

Let B_n be the n -disk in \mathbb{H}^2 with the center origin O . Let $\mathbf{B}_n = B_n \times [-n, n]$ be the solid cylinder with height $2n$ and radius n . Let $\mathcal{C}_n = \partial \mathbf{B}_n$ be the cylinder with caps.

Let π_n be the radial projection from $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ into \mathcal{C}_n which maps the corners of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ into corners of \mathcal{C}_n . Then, define $\gamma_n = \pi_n(\Gamma)$ which is a collection of Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Notice that \mathbf{B}_n is convex by construction. Then, by solving the Plateau problem in \mathbf{B}_n for γ_n , we get area minimizing surfaces Σ_n in \mathbf{B}_n with $\partial \Sigma_n = \gamma_n$ [Fe]. Notice that since \mathbf{B}_n is a convex domain in $\mathbb{H}^2 \times \mathbb{R}$, Σ_n is smooth, and it is area minimizing not only in \mathbf{B}_n , but also in $\mathbb{H}^2 \times \mathbb{R}$.

Let $\Lambda = \mathbb{H}^2 \times \mathbb{R} - (\mathcal{N}^+ \cup \mathcal{N}^-)$. Then, $\partial_\infty \Lambda \cap (\mathbb{H}^2 \times \{\pm\infty\}) = \Gamma^+ \cup \Gamma^-$. Hence, by modifying γ_n if necessary, we will assume that $\gamma_n \subset \Lambda \cap \partial \mathbf{B}_n$ and $\gamma_n \rightarrow \Gamma$. Again, Σ_n is the area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial \Sigma_n = \gamma_n$.

The Limit of the sequence: We claim that $\Sigma_n \subset \Lambda$. Indeed, our proof works not only for area minimizing surfaces, but also minimal surfaces. In other words, we will show that if $\gamma_n \subset \Lambda$ and S_n is a *minimal surface* with $\partial S_n = \gamma_n$, then $S_n \subset \Lambda$.

By construction, if we show that $\Sigma_n \cap \mathcal{N}^\pm = \emptyset$, we are done. Recall that $\mathcal{N}^+ = \bigcup_{i=1}^{n_1} \mathcal{N}_i^+ \bigcup_{j=1}^{2n} \widehat{\mathcal{T}}_j^+$. First, we consider infinite side barriers $\widehat{\mathcal{T}}^\pm$. In particular, $\widehat{\mathcal{T}}_j^+ = \bigcup \widehat{\mathcal{T}}_{jt}^+$ where $\widehat{\mathcal{T}}_{jt}^+$ is bounded by an infinite rectangle. We

claim that if $\gamma_n \cap \widehat{\mathcal{T}}_{jt}^+ = \emptyset$, then $\Sigma_n \cap \widehat{\mathcal{T}}_{jt}^+ = \emptyset$. Recall that $\widehat{\mathcal{T}}_{jt}^+$ is the domain in $\mathbb{H}^2 \times \mathbb{R}$ separated by the area minimizing surface \mathcal{T}_{jt}^+ where $\partial_\infty \mathcal{T}_{jt}^+$ is an infinite rectangle \mathcal{R}_{jt}^+ . Then, $\widehat{\mathcal{T}}_{jt}^+$ is foliated by the vertical translations of \mathcal{T}_{jt}^+ . Since $\partial \Sigma_n \cap \widehat{\mathcal{T}}_{jt}^+ = \emptyset$, if $\Sigma_n \cap \widehat{\mathcal{T}}_{jt}^+ \neq \emptyset$, then the last point of touch with the minimal foliation will give a contradiction by the maximum principle. This implies $\Sigma_n \cap \widehat{\mathcal{T}}_{jt}^+ = \emptyset$, and hence $\Sigma_n \cap \widehat{\mathcal{T}}_j^+ = \emptyset$.

Now, we consider Scherk barriers. Recall that $\mathcal{N}_i^+ = \bigcup \mathcal{N}_{ik}^+$ where $\partial \mathcal{N}_{ik}^+ \subset \widehat{S}_{ik}^t \cup \partial \widehat{\mathcal{T}}^+$. Recall that \widehat{S}_{ik}^t is a subsurface of the Scherk graph S_{ik}^t , and the vertical translations of \widehat{S}_{ik}^t foliates \mathcal{N}_{ik}^+ . Notice that $\Sigma_n \cap \widehat{\mathcal{T}}^+ = \emptyset$ and $\gamma_n \cap \mathcal{N}_{ik}^+ = \emptyset$. Hence, if $\Sigma_n \cap \mathcal{N}_{ik}^+ \neq \emptyset$, then the last point of touch with the minimal foliation must be in the interior of Σ_n . This gives a contradiction by the maximum principle as before. This shows that $\Sigma_n \cap \mathcal{N}_i^+ = \emptyset$.

Now, $\Sigma_n \cap \mathcal{N}_i^+ = \emptyset$ and $\Sigma_n \cap \widehat{\mathcal{T}}_j^+ = \emptyset$ together implies $\Sigma_n \cap \mathcal{N}^+ = \emptyset$. Hence, if $\gamma_n \subset \Lambda$, then $\Sigma_n \subset \Lambda$ as claimed.

In particular, this implies that \mathcal{N}^\pm is indeed a barrier for the sequence $\{\Sigma_n\}$. By compactness theorem of geometric measure theory [Fe], the sequence of area minimizing surfaces $\{\Sigma_n\}$ has convergent subsequences in compact sets B_n . Hence, by using the diagonal sequence argument, we obtain a limit area minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, $\Sigma \subset \Lambda$ as $\Sigma_n \subset \Lambda$ for any n . This shows that $\partial_\infty \Sigma \subset \partial_\infty \Lambda$. As in [Co1, Theorem 2.14], for the cylinder part $\widetilde{\Gamma}$, we can use tall rectangles as barriers. This proves that $\partial_\infty \Sigma \subset \Gamma$. Since $\partial \Sigma_n = \gamma_n \rightarrow \Gamma$, by using the linking argument in [Co1], we conclude that $\partial_\infty \Sigma = \Gamma$. Step 1 follows. \square

Step 2: If Γ is strongly fillable, then Γ is fat at infinity.

Proof of Step 2: Let Γ be strongly fillable curve, i.e. $\partial_\infty \Sigma = \Gamma$ where Σ is area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$. Recall that as indicated in Remark 2.15, we omit the infinite curves neither skinny nor fat. So, we will assume Γ is skinny, and get a contradiction.

Assume that Γ is skinny. Without loss of generality, assume $\Gamma^+ = \gamma^1 \cup \dots \cup \gamma^k$ induces an ideal $2k$ -gon Δ where $a(\Delta) > b(\Delta)$. If Δ decomposes into more than one inscribed polygons, take Δ as the skinny polygon in the decomposition.

Let Σ be an area minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$. Then, consider the sequence $S_n = \Sigma - n$ which is vertical translation down by n . By construction, the limit of the sequence $\{S_n\}$ is the collection of vertical geodesic planes $\Sigma^+ = \Gamma^+ \times \mathbb{R}$ (Lemma 2.7). Furthermore, as the limit of area minimizing surfaces is area minimizing, Σ^+ is also area minimizing.

Let $\partial \Delta = \gamma^1 \cup \beta^1 \cup \dots \cup \gamma^k \cup \beta^k$. By assumption $a(\Delta) > b(\Delta)$. Here, $a(\Delta)$ corresponds to total "length" of $\Gamma^+ = \gamma^1 \cup \dots \cup \gamma^k$, and $b(\Delta)$ corresponds

to the total "length" of the remaining geodesics in $\partial\Delta$, i.e. $\partial\Delta^+ - \Gamma^+ = \beta^1 \cup \dots \beta^k$ (See Section 2.3).

Recall that B_m be the disk of radius m and center O in \mathbb{H}^2 , and $\mathbf{B}_m = B_m \times [-m, m]$ is the solid cylinder in $\mathbb{H}^2 \times \mathbb{R}$. Consider $\Sigma_m^+ = \mathbf{B}_m \cap \Sigma^+$. We claim that for sufficiently large m , Σ_m^+ is not an area minimizing surface. Let $\eta_m = \partial\Sigma_m^+$ be the collection of disjoint k Jordan curves in $\partial\mathbf{B}_m$.

For $1 \leq i \leq k$, let $\gamma_m^i = \gamma^i \cap B_m$. In other words, γ_m^i is a finite arc segment in the infinite geodesic γ^i . Then, $\Sigma_m^+ = \bigcup \gamma_m^i \times [-m, m]$ is a collection of k vertical geodesic surfaces in \mathbf{B}_m by construction. Let $\partial\gamma_m^i = \{p_m^{2i-1}, p_m^{2i}\}$ be the endpoints of γ_m^i . Then, we have $2k$ points $\mathcal{V}_m = \{p_m^1, p_m^2, \dots, p_m^{2k}\}$ in ∂B_m . Let β_m^i be the geodesic in B_m connecting p_m^{2i} and p_m^{2i+1} . Hence, γ_m^i and β_m^i curves defines a polygon Δ_m in B_m , i.e. $\partial\Delta_m = \gamma_m^1 \cup \beta_m^1 \cup \dots \gamma_m^k \cup \beta_m^k$.

Now, let $\Pi_m^i = \beta_m^i \times [-m, m]$ be a vertical geodesic surface in \mathbf{B}_m . Let $\Delta_m^+ = \Delta_m \times \{m\}$ and $\Delta_m^- = \Delta_m \times \{-m\}$, i.e. $\Delta_m^\pm \subset \partial\mathbf{B}_m$. Then, define a surface $S_m = \bigcup_{i=1}^k \Pi_m^i \cup \Delta_m^+ \cup \Delta_m^-$ in \mathbf{B}_m . S_m is topologically a sphere with k holes. Furthermore, $\partial S_m = \partial\Sigma_m = \eta_m$.

We claim that the area of S_m is less than the area of Σ_m for sufficiently large m . Let $\|\cdot\|$ and $|\cdot|$ represent the area and the length respectively. Since Δ is an ideal $2k$ -gon, then $\|\Delta\| = 2(k-1)\pi$. By construction, $\Delta_m^\pm \subset \Delta$ for any m . Hence, $\|\Delta_m^\pm\| < 2(k-1)\pi$.

Now, by assumption $a(\Delta) > b(\Delta)$. By the definition of $a(\cdot)$ and $b(\cdot)$, this implies that $a_m = \sum_{i=1}^k |\gamma_m^i| > \sum_{i=1}^k |\beta_m^i| = b_m$ for sufficiently large m . Let $c_m = a_m - b_m$ for m large, and let $c = a(\Delta) - b(\Delta) > 0$. Then, $c_m \nearrow c$ as $m \rightarrow \infty$, and hence $c_m > 0$ for sufficiently large m . Now, $\|\Sigma_m^+\| = 2m \cdot a_m$ and $\|S_m\| < 2m \cdot b_m + 4(k-1)\pi$. Since $c_m \nearrow c$, for sufficiently large m , $\|\Sigma_m^+\| > \|S_m\|$. This proves that Σ_m^+ is not an area minimizing surface, and give a contradiction. Step 2, and hence the proof of the theorem follows. \square

4. FILLABLE AND NON-FILLABLE INFINITE CURVES

In this part, we will see that fillability question and strongly fillability question are quite different for infinite curves. In particular, for a given infinite tall curve Γ , while Γ^\pm completely determines strong fillability of Γ , Γ^\pm is not very useful to detect whether Γ is fillable.

4.1. Γ^\pm and Fillability.

A trivial observation is that for any given collection of disjoint geodesics $\gamma_1 \cup \dots \cup \gamma_n$ in $\mathbb{H}^2 \times \{+\infty\}$, there is a fillable curve Γ with $\Gamma^+ = \gamma_1 \cup \dots \cup \gamma_n$. In particular, if $\mathcal{P}_i = \gamma_i \times \mathbb{R}$ is the vertical plane over γ_i , then $S = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_2$ is a collection of minimal planes, and $\Gamma = \partial_\infty S$ would be a fillable curve with $\Gamma^+ = \gamma_1 \cup \dots \cup \gamma_n$. So, only Γ^+ (or only Γ^-) is not enough to determine if Γ is fillable or not.

Furthermore, we will show that knowing both Γ^+ and Γ^- together is not enough to determine whether Γ is fillable or not. First, by using the following theorem, we have a very large family of fillable curves.

Theorem 4.1. *Let Γ be a curve with n components in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, i.e. $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$. If each Γ_i is tall, and fat at infinity, then Γ is fillable.*

Proof: By Theorem 3.1, there exists an area minimizing surface Σ_i with $\partial_\infty \Sigma_i = \Gamma_i$. Let $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$. Therefore, if $\Sigma_i \cap \Sigma_j = \emptyset$, then Σ would be an embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, and we are done.

By assumption $\Gamma_i \cap \Gamma_j = \emptyset$. Since $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is topologically a sphere, Γ_i and Γ_j bounds disjoint regions in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, i.e. $\Gamma_i = \partial \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$. Then by [Co1, Lemma 2.19], the area minimizing surfaces bounding such disjoint curves are disjoint, i.e. $\Sigma_i \cap \Sigma_j = \emptyset$.

This shows that Σ is a collection of disjoint area minimizing surfaces. Therefore, Σ is an embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$, and Γ is fillable. The proof follows. \square

Remark 4.2. [Fillable but not Strongly Fillable Curves] Note that the theorem above does not show that such a Γ is strongly fillable. This is because the union of area minimizing surfaces may not be area minimizing. For example, if Γ^+ is skinny at infinity, the surface $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$ is not area minimizing by Theorem 3.1, even though each Σ_i is area minimizing. However, it is still a minimal surface, and hence Γ is fillable.

By using this idea, it is easy to construct many examples of fillable, but not strongly fillable curves. In particular, by choosing a collection of Jordan curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ where Γ is skinny at infinity, then Γ is fillable, but not strongly fillable by Theorem 3.1 and Theorem 4.1.

By using this theorem and the infinite rectangles (Lemma 2.9), we show that for any given collection of geodesics Γ^+ and Γ^- , it is possible to construct a fillable curve Γ .

Corollary 4.3. *Let $\gamma_1^+ \cup \dots \cup \gamma_n^+$ and $\gamma_1^- \cup \dots \cup \gamma_m^-$ be given collections of disjoint geodesics in $\mathbb{H}^2 \times \{+\infty\}$ and $\mathbb{H}^2 \times \{-\infty\}$ respectively. Then, there exists a fillable curve Γ with $\Gamma^+ = \gamma_1^+ \cup \dots \cup \gamma_n^+$ and $\Gamma^- = \gamma_1^- \cup \dots \cup \gamma_m^-$.*

Proof: Let $\partial_\infty \gamma_i^+ = \{p_i^+, q_i^+\}$, and α_i^+ be the shorter arc in S_∞^1 with the endpoints $\{p_i^+, q_i^+\}$. Let t_1, t_2, \dots, t_n be real numbers such that if $\alpha_i^+ \subset \alpha_j^+$, then $t_i > t_j$. Then, let $\hat{\alpha}_i^+ = \alpha_i^+ \times \{t_i\}$ be an arc in $S_\infty^1 \times \{t_i\}$. Then let \mathcal{R}_i^+ be the infinite rectangle with $\partial_\infty \mathcal{R}_i^+ \supset \gamma_i^+ \cup \hat{\alpha}_i^+$. Similarly, define \mathcal{R}_j^- . As both $\{\gamma_i^+\}$ and $\{\gamma_j^-\}$ are pairwise disjoint family of geodesic arcs, and both $\{\hat{\alpha}_i^+\}$ and $\{\hat{\alpha}_j^-\}$ are arcs in different levels $S_\infty^1 \times \{t_i\}$, then $\Gamma = \bigcup_i \mathcal{R}_i^+ \bigcup_j \mathcal{R}_j^-$ is a union of pairwise disjoint infinite rectangles (See Figure 6-left).

Let \mathcal{T}_i^+ be the unique area minimizing surface \mathcal{R}_i^+ bounds, and \mathcal{T}_j^- be the unique area minimizing surface \mathcal{R}_j^- bounds. Then, $\mathcal{S} = \bigcup_i \mathcal{T}_i^+ \bigcup_j \mathcal{T}_j^-$ is a union of pairwise disjoint minimal surfaces by Theorem 4.1. Hence, \mathcal{S} is a complete embedded minimal surface with $\partial_\infty \mathcal{S} = \Gamma$. Hence, Γ is fillable, and by construction $\Gamma^+ = \gamma_1^+ \cup \dots \cup \gamma_n^+$ and $\Gamma^- = \gamma_1^- \cup \dots \cup \gamma_m^-$. The proof follows. \square

Remark 4.4. Notice again that the fillable Γ constructed above may not be strongly fillable by Remark 4.2. Furthermore, even if Γ is fat at infinity, we cannot conclude that the surface \mathcal{S} constructed in the corollary above is area minimizing. This is because the area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$ may be connected, or some other surface in $\mathbb{H}^2 \times \mathbb{R}$. In any case, \mathcal{S} is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \mathcal{S} = \Gamma$.

4.2. Examples of Non-fillable Curves.

First, of course, if Γ^\pm is not a collection of geodesics, then by Lemma 2.7, Γ is not fillable. Therefore, we only consider the nontrivial case that Γ^\pm is a collection of geodesics.

Even though the previous section shows that Γ^\pm fails to detect fillability of Γ , in some cases, with some conditions on $\tilde{\Gamma}$, it shows that Γ is non-fillable. In this part, we will give a family of examples of non-fillable infinite curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.

Let ξ be a Scherk curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ (Remark 2.12). Let $\Delta_\xi = \Delta_\xi^+ = \Delta_\xi^-$ be ideal polygon in $\mathbb{H}^2 \times \{\pm\infty\}$ induced by ξ^\pm (Section 2.4). Let $\hat{\Delta}_\xi^+$ be the component of $\mathbb{H}^2 \times \{+\infty\} - \xi^+$ containing Δ_ξ^+ . Similarly, define $\hat{\Delta}_\xi^-$.

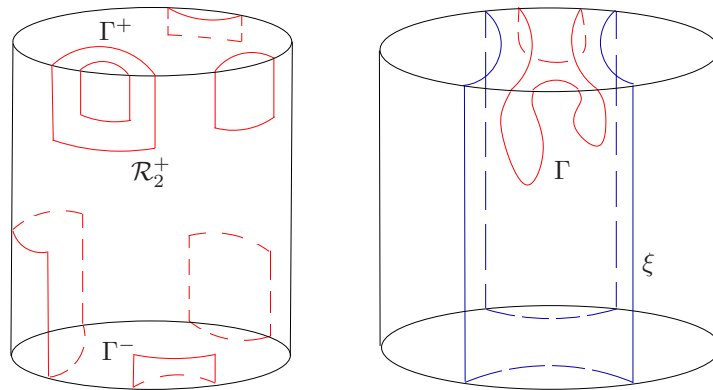


FIGURE 6. In the figure left, given $\Gamma^+ = \gamma_1^+ \cup \dots \cup \gamma_4^+$ and $\Gamma^- = \gamma_1^- \cup \dots \cup \gamma_3^-$, we construct fillable Γ with infinite rectangles \mathcal{R}_i^\pm for each γ_i^\pm . In the figure right, ξ is a Scherk curve, and Γ is trapped by ξ .

Definition 4.5 (Trapped Curves). Let Γ be an infinite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Assume that there exists a Scherk curve ξ such that $\Delta_\Gamma^+ \subset \widehat{\Delta}_\xi^+$. If $\Gamma \cap \xi = \emptyset$ and $\Delta_\Gamma^+ \cap \Delta_\xi \neq \emptyset$, then we will call Γ *trapped by ξ* with notation $\Gamma \prec \xi$. Similarly, extend the definition to the curves by replacing all $+$ signs with $-$ signs in the corresponding places. See Figure 6-right.

Theorem 4.6. *Let Γ be an infinite curve trapped by a Scherk curve ξ . Then, Γ is not fillable.*

Proof: The proof is a straightforward application of the maximum principle. Assume that there exists a minimal surface Σ with $\partial_\infty \Sigma = \Gamma$. Let S_0 be a Scherk graph with $\partial_\infty S_0 = \xi$. Parametrize all Scherk graphs bounding ξ such that for $t \in \mathbb{R}$, $S_t = S_0 + t$ vertical translation of S_0 by t . Then, for any t , $\partial_\infty S_t = \xi$, and the family $\{S_t\}$ foliates the convex region $\Delta_\xi \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$.

Without loss of generality, assume $\Delta_\Gamma^+ \subset \widehat{\Delta}_\xi^+$. Then, for sufficiently large $t_o > 0$, $S_{-t_o} \cap \Sigma = \emptyset$ since $S_{-t} \rightarrow \xi^+ \times \mathbb{R}$ as $t \rightarrow \infty$ and $\Gamma \cap \xi = \emptyset$. As $\Delta_\Gamma^+ \cap \Delta_\xi \neq \emptyset$, $\Sigma \cap (\Delta_\xi \times \mathbb{R}) \neq \emptyset$. Then, let $t_1 = \inf\{t \mid S_t \cap \Sigma \neq \emptyset\}$. Then, being the first point of touch, S_{t_1} intersects Σ tangentially with lying in one side. This contradicts to the maximum principle. Similar arguments work for $\Delta_\Gamma^- \subset \widehat{\Delta}_\xi^-$ case, too. The proof follows. \square

5. FINAL REMARKS

5.1. Con-Fillable Curves.

As indicated in Section 4.1, the behavior of Γ^\pm is highly inadequate to detect fillability of an infinite curve. On the other hand, the main result, Theorem 3.1, shows that Γ^\pm completely determines the strong fillability of an infinite curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Notice that the minimal surfaces constructed in Corollary 4.3 for a given Γ^\pm are collection of disjoint minimal surfaces for each component in Γ^\pm . Hence, the following natural question becomes very interesting: *If we restrict connectedness on Γ , or the filling minimal surface, does Γ^\pm still plays a crucial role to determine fillability of Γ ?*

This question suggests the following notion between fillability and strong fillability. We will call a curve Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *con-fillable*, if the filling minimal surface Σ is connected. In other words, Γ is con-fillable if there exists a connected, complete, embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$.

Question 1: [Con-fillability] Which infinite curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ bounds a connected, complete, embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$?

On the other hand, we can state a simpler version of this question as follows:

Question 1a: Let Γ be an infinite Jordan curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Is Γ fillable?

Notice that in the other versions of the problem, we assume Γ to be a finite collection of disjoint Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. If we assume Γ to be one Jordan curve, then the filling surface would automatically be connected. Hence, Question 1a is just a simpler case of Question 1. On the other hand, by changing Question 1a slightly, "Which Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ bounds a minimal (or least area) plane in $\mathbb{H}^2 \times \mathbb{R}$?" is another interesting question where the analogous question for \mathbb{H}^3 was studied by [An, Co2].

Both of these questions are very interesting as the examples constructed in Theorem 4.3 do not apply to these cases. Hence, Γ^\pm being fat or skinny might play a crucial role to detect con-fillability of the curve Γ .

On the other hand, con-fillability and strong fillability are very different notions, where one does not include the other one. In particular, we will give two families of examples of curves which are only con-fillable, and only strongly fillable.

Con-fillable, but not strongly fillable curves:

Horizontal catenoids S_2 and minimal k -noids S_k constructed in [MoR, Py] give an important family of examples for con-fillable curves for any $k \geq 2$. Notice that the asymptotic boundary of horizontal catenoids and minimal k -noids consists of k infinite "vertical" Jordan curves, i.e. $\Gamma_k = \partial_\infty S_k = \gamma_1 \cup \dots \cup \gamma_k$ where γ_i is the asymptotic boundary of a vertical geodesic plane in $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, by construction, Γ_k is skinny at infinity for any $k \geq 2$. This shows Γ_k does not bound any area minimizing surface by Theorem 3.1. Hence, these are also examples of con-fillable, but not strongly fillable curves.

Strongly fillable, but not con-fillable curves:

For a given two disjoint geodesics τ_1 and τ_2 in \mathbb{H}^2 , define $\Gamma_i = \partial_\infty(\tau_i \times \mathbb{R})$. By [MoR], there is a constant $\eta_0 > 0$ such that if $d(\tau_1, \tau_2) < \eta_0$ then $\Gamma = \Gamma_1 \cup \Gamma_2$ bounds a horizontal catenoid in $\mathbb{H}^2 \times \mathbb{R}$. On the other hand, if $d(\tau_1, \tau_2) > \eta_0$, the curve Γ is an example of a strongly fillable curve, which is not con-fillable. In order to see this, first notice that $\Sigma = (\tau_1 \cup \tau_2) \times \mathbb{R}$ is an area minimizing surface by Theorem 3.1. Now, assume that there is a connected minimal surface T with $\partial_\infty T = \Gamma$. By $d(\tau_1, \tau_2) > \eta_0$, there is a Scherk curve β in the region between Γ_1 and Γ_2 in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with $\beta \cap \Gamma = \emptyset$ by [MoR]. By translating the Scherk graph S , we can assume that $S \cap T = \emptyset$. However, as T is connected, a vertical translation of S must intersect T . This implies first point of touch gives a contradiction with the maximum principle as in Theorem 4.6.

5.2. Finite Curves.

In [Co1], we discussed the asymptotic Plateau problem for finite curves, and give a fairly complete classification for strongly fillable curves. In this paper, we completed this classification by giving a characterization for infinite strongly fillable curves.

While strong fillability question has been finished, fillability question for finite and infinite curves are wide open. In particular, we gave examples of fillable and non-fillable curves in Section 4. Furthermore, the same question for finite curves is also very delicate. By a simple generalization of Theorem 4.1, if Γ is a collection of strongly fillable (finite or infinite) curves, then it is fillable. On the other hand, the only known family of finite non-fillable curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is the curves containing thin tails by Lemma 2.4. However, there are many curves Γ with $h(\Gamma) < \pi$ containing no thin tails. In [Co1, KM], some families of fillable examples, namely butterfly curves, have been constructed. However, it is still wide open question that which curves are fillable among such curves?

Similarly, con-fillability question is also wide open for finite curves, too. *Which curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ bounds a connected minimal surface?* As discussed in previous section, this question is also very different from the fillability question, and the strong fillability question. In [FMMR], the authors studied a special case of this problem, namely for minimal annuli. By using the ideas for not con-fillable examples above, it is not hard to show that a finite curve where the components are "horizontally apart", then it is not con-fillable.

In particular, if $\Gamma = \gamma_1 \cup \dots \cup \gamma_n$ is a finite curve where Γ stays in one side of a Scherk curve, then Γ can not con-fillable. Similarly, if a finite curve is "vertically π -apart", then it is not con-fillable. Again, if $\Gamma = \gamma_1 \cup \dots \cup \gamma_n$ is a finite curve where some γ_i is in $S_\infty^1(\mathbb{H}^2) \times (-\infty, t)$ and all other γ_j is in $S_\infty^1(\mathbb{H}^2) \times (t + \pi, \infty)$, then Γ cannot be con-fillable. This is because for the pair of horizontal circles $\alpha_\epsilon = S_\infty^1(\mathbb{H}^2) \times \{t + \epsilon, t + \pi - \epsilon\}$ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, there is a catenoid \mathcal{C}_ϵ with $\partial_\infty \mathcal{C}_\epsilon = \alpha_\epsilon$ such that $S \cap \mathcal{C}_\epsilon = \emptyset$ for a given minimal surface S with $\partial_\infty S = \Gamma$. Then, by using horizontal hyperbolic translations φ_t of \mathcal{C}_ϵ , we can get a contradiction with maximum principle at the first point of contact of $\varphi_t(\mathcal{C}_\epsilon)$ and S .

Let Γ be *semi-infinite* if only one of Γ^+ or Γ^- is nonempty. Fillability, and con-fillability questions for a semi-infinite curve Γ might be detected by the behavior at infinity. For example, "Are there any fillable semi-infinite Jordan curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which is skinny at infinity?" seems an interesting question, and easier case to study for fillability question. By considering Theorem 4.6, if there exists such a curve, it is a curve not trapped by a Scherk curve. While Γ^\pm hardly detects fillability for infinite curves by section 4.1, it might be the key property for semi-infinite curves.

5.3. Asymptotic H -Plateau Problem in $\mathbb{H}^2 \times \mathbb{R}$.

Constant Mean Curvature (CMC) surfaces are natural generalizations of minimal surfaces. In \mathbb{H}^3 and $\mathbb{H}^2 \times \mathbb{R}$, many analogous questions have been studied in CMC case, too. Like $H \in [0, 1)$ for \mathbb{H}^3 , $H \in [0, \frac{1}{2})$ is the interesting case for complete embedded H -surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

We call a curve Γ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ *H -fillable* if there exists an H -surface Σ_H in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma_H = \Gamma$. Hence, the following generalization is very natural:

Which curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ is H -fillable for $H \in [0, \frac{1}{2})$?

Note that this question has been discussed in [NSST]. In particular, by [NSST, Theorem 4.1], an H -fillable curve Γ cannot be finite or semi-infinite. Furthermore, $\tilde{\Gamma}$ must be a collection of vertical lines. On the other hand, very different from the $H = 0$ case, for $H > 0$ case, there are infinite H -catenoids, and H -paraboloids in $\mathbb{H}^2 \times \mathbb{R}$ [NSST]. These surfaces make the question very different from the usual asymptotic Plateau problem. Furthermore, because of these a priori properties for H -fillable curves mentioned above, and the existence of these H -surfaces, it might be a better idea to study this question in the geodesic compactification of $\mathbb{H}^2 \times \mathbb{R}$ [KM] rather than the product compactification.

Note also that Scherk graphs described in Section 2.3 were generalized to CMC context, say Scherk H -graphs, for $H \in [0, \frac{1}{2})$ by [HRS]. Hence by replacing Scherk graphs with these Scherk H -graphs, it might possible to define the fat/skinny at infinity notions for curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, and generalize a version of Theorem 3.1 to CMC case by following similar ideas.

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